MATH 310
Indicative problems for the final exam

1. Find all solutions to the linear system:

$$\left.\begin{array}{rcl} 2x_1 - -2x_2 + 7x_3 + 7x_4 & = & 1 \\ -x_1 + x_2 - 2x_3 + x_4 & = & -2 \\ x_1 - x_2 + 3x_3 + 2x_4 & = & 1 \end{array}\right\}$$

2. Let $V = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 | 2x_1 + x_2 - x_3 = 0\}$. Show that V is a subspace of $\mathbb{R}^3$ and find a basis for it.

3. Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear map:

$$L((x_1, x_2, x_3)^T) = (x_1 - x_2 + x_3, x_1 - 2x_2, x_2 - 3x_3, 2x_1 + x_3)^T$$

   Find the matrix of L with respect to the following bases:
   $\{(0,0,1)^T, (0,1,1)^T, (1,1,1)^T\}$ for $\mathbb{R}^3$ and
   $\{(0,1,0,0)^T, (1,1,0,0)^T, (0,0,1,0)^T, (0,0,1,1)^T\}$ for $\mathbb{R}^4$.

4. Find a basis for the nullspace, the row - space and the column - space of the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 2 & 5 & 2 \\ -1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

5. Compute the following determinants using any method you like:

$$\begin{vmatrix} 1 & 4 & -2 \\ 2 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & -1 & 1 \\ 3 & 2 & -1 & 2 \\ -1 & 2 & 1 & 3 \\ 1 & 1 & 3 & 2 \end{vmatrix}$$

6. Find the transition matrix from the basis:
   $\{(1,-1,1)^T, (1,0,0)^T, (0,1,1)^T\}$
   to the basis:
   $\{(0,0,1)^T, (0,1,1)^T, (0,0,1)^T\}$
   and conversely.

7. Use the explicit formula for the inverse of a matrix in order to compute the inverse of

$$\begin{pmatrix} 2 & -3 & -2 \\ 2 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

8. Use Cramer's rule in order to solve the system:

$$\left.\begin{array}{rcl} 3x - y + 2z & = & 1 \\ x - 2y + z & = & 3 \\ 2x + 3y + z & = & -1 \end{array}\right\}$$

9. Find the rank and the nullity of the matrix:

$$\begin{pmatrix} 1 & -3 & -2 & -2 \\ -1 & 1 & -1 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix}$$

10. Find a basis for the column space and a basis for the row space of the matrix:

$$\begin{pmatrix} 1 & -2 & -2 \\ -1 & -1 & 3 \\ 2 & 1 & 3 \\ 1 & -3 & 2 \end{pmatrix}$$

11. Find a linear map whose matrix with respect to some bases is:

$$\begin{pmatrix} 1 & -3 & -2 \\ -1 & 1 & -1 \end{pmatrix}$$

Deterrmine a basis for its kernel and a basis for its image.

12. Consider the following bases of $\mathbb{R}^3$.

$$u_1 = (1,1,1)^T \quad u_2 = (1,1,0)^T \quad u_3 = (1,0,0)^T$$

$$v_1 = (1,0,1)^T \quad v_2 = (1,1,0)^T \quad v_3 = (0,1,1)^T$$

Determine the transition matrix from the 1st to the 2nd and the one from the 2nd to the 1st.

13. Consider the map $L : \mathbb{R}^3 \to \mathbb{R}^4$ with

$$L((x_1,x_2,x_3)^T) = (2x_1 - x_2, x_2 + 3x_3, 2x_1 + 3x_3, 2x_1 - 2x_2 - 3x_3)^T$$

Find orthonormal bases for its kernel and for its image.

14. Find a subspace V of $\mathbb{R}^3$, so that $\mathbb{R}^3$ is the direct sum of the subspace U which is spanned by $(1,1,1)^T$ and V.

Let U be the subspace of $\mathbb{R}^3$ generated by $(1,1,0)$ and $(1,-1,0)$. Write $(3-1,5)$ as the sum of a vector that belongs to U and a vector that belongs to $U^\perp$.

15. Let U be the subspace of $\mathbb{R}^4$ spanned by $(1,1,-1,1)^T$ and $(1,0,1,0)^T$. Find an orthonormal basis for $U^\perp$.

16. Consider the 2×2 matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Find a third column to add on the right of this matrix so that it becomes orthogonal. How many such columns can you find?

17. Let A be the matrix:

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -2 & 1 \end{pmatrix}$$

Determine a basis for the orthogonal complement of $N(A)$.

18. Find the least squares solution of the system:

$$\left.\begin{array}{rcl} x - y & = & 1 \\ x - 2y & = & 3 \\ 2x + 3y & = & -1 \end{array}\right\}$$

19. Find the QR-factorization of the matrix:

$$\begin{pmatrix} 1 & -2 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$$

Find the eigenvalues and the corresponding eigenspaces of the matrix:

$$\begin{pmatrix} 7 & -24 \\ 2 & -7 \end{pmatrix}$$

20. Find a matrix that diagonalizes:

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

21. Show that the matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 7 \end{pmatrix}$$

is positive definite.