Analysis of the Prime-Number Theorem
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Outline of Selberg’s Proof

To summarize Selberg’s proof of the prime number theorem, Selberg divides the proof into four different sections. In the first section, Selberg introduces the prime number theorem as:

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1 \text{ for } x > 0$$

He also defines the prime-number theorem through summation. This can be written as:

$$\theta(x) = \sum_{p \leq x} \log p$$

Generally speaking, Selberg invokes multiple definitions and formulas deduced from this form of the prime-number theorem and goes on to prove each individual part. He also makes a small side note that his first proof consisted of using the results from Erdős. One side note to mention: throughout this proof, $p$, $q$, and $r$ are all denoted as prime numbers. As a result of all these basic formulas, he establishes that:

$$\theta(x) \log x + \sum_{p \leq x} \theta\left(\frac{x}{p}\right) \log p = 2x \log x + O(x)$$

In the second section of Selberg’s prime number theorem, Selberg defines basic formulas and proves them. He applies summation and property of logarithms in order to expand the definition of the prime number theorem. Then, connecting each of the equations that he deduced to, he formulated the equation:

$$|R(x)| \leq \frac{1}{\log x} \sum_{n \leq x} |R\left(\frac{x}{n}\right)| + O\left(\frac{x \log \log x \log x}{\log x}\right)$$

In the third part of the proof, Selberg defines some properties of $R(x)$ that will help develop the proof even further. In this case:

$$R(x) = \theta(x) - x$$

Overall, using $R(x)$, Selberg establishes the existence of a constant $k > 0$ so that every $\delta > 0$ and $x > e^{k/\delta}$ the interval $(x, xe^{k/\delta})$ contains subintervals $(y, ye^{\delta/2})$ inside which

$$|R(z)| < 4\delta z$$

In the final section of the proof, with all the pieces put together, Selberg proves the prime number theorem by using the bootstrapping argument on how to estimate $|R(x)| < ax$. This equation can be improved to the follow:

$$|R(x)| < a \left(1 - \frac{a^2}{300k^2}\right) x \text{ if we go far enough.}$$

This is enough information to prove the prime number theorem according to Selberg. Later in this paper, we will go over one part of the Selberg’s proof and how he was able to establish the equation.
History on Selberg

Before we go into detail of Selberg’s proof of the prime number theorem, we will introduce who the well-known mathematician is.

Atle Selberg was born on June 14, 1917 in Langesund, Norway. His mother was Anna Kristina Brigtsdatter Skeie and Ole Michael Ludvigsen Selberg. Selberg’s dad was a math teacher at the University of Oslo. When he was 48, he earned his doctorate for his thesis “Ein Beitrag zur Theorie der algebraisch auflösbaren Gleichungen von Primzahligrad” (which is roughly translated to “A Contribution to the Theory of Algebraic Equations of Prime Degree”). Selberg’s mother was also a teacher. Atle Selberg was the youngest of 9 children (5 boys and 4 girls). Three of Atle’s brothers also became mathematicians.

Because of his dad’s job, Selberg grew up in Voss, Bergen, and Gjovik. Ironically, Selberg did not learn mathematics through his father. Instead, he learned through the mathematics books that his father kept in a library. In middle school, Selberg taught himself methods to solve different equations through the books that he read. One example he came up with was:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^n}{2n + 1} \text{ where } n \geq 0$$

When he came across equation years later, he said “... [it was] such a very strange and beautiful relationship that I determined that I would read that book in order to find out how that formula came about.” As he continued to read, the lecture notes introduced real numbers using Dedekind cuts (which is a set partition of rational numbers split into two nonempty subsets $S_1$ and $S_2$ where $S_1 < S_2$ and $S_1$ has no greatest member). He was baffled by this introduction but found the notes inspiring. (Pictured below is an example of Dedekind cuts)

This was his main interest at the time, but one of his brothers convinced him otherwise. While Selberg was in high school, one of his brother suggested that he reads about Chebyshev’s work on the distribution of primes. Selberg came across Ramanujan’s work and was impressed by the mathematics. Because of this, Selberg went on to study number theory and began to make his own mathematical discoveries.

In 1935, Selberg graduated from high school and attended the University of Oslo. While there, his brother Henrik introduced him to Stormer, one of the mathematicians that influenced him to go into mathematics. He was also influenced by a lecture by Erich Hecke at the International Mathematical Conference in Oslo in 1936. In spring of 1939, he graduated with a master’s degree in mathematics. Following up with his graduation, he completed his first part of his military service. With this, he planned to travel to Hamburg to research with Erich Hecke. He applied and received a scholarship to fund for the trip but plans changed when World War II began.

Instead of traveling to Hamburg, he went to Uppsala and attended Trygve Nagell’s lectures to help with his doctoral research. When he returned to Oslo on December of 1939, he stated the following: “...the war came to Norway at the beginning of April 1940, and that caused an interruption of my mathematical research. I did not think about mathematics while I fought with the Norwegian forces against the German invaders in Gudbrandsdalen. We were also in the upper part of Osterdalen, and ended up near Andalsnes. I was a
soldier in Major Hegstad’s artillery battalion, and I held him in very high esteem. Also [I did not think about mathematics] while I was a prisoner of war at the prison camp at Trandum. When I finally was released, I travelled to the west coast of Norway, and later with my family to Hardanger.” Through the struggles of World War II, in 1943, he defended his thesis on the zeros of Riemann’s zeta-function. Then the German invaders swarmed in and arrested Selberg. However, he was allowed to be released so long as he left Oslo and return to Gjovik with his parents. So because of this, he researched on the Riemann hypothesis on his own back at home.

Some notes about Selberg’s personal life: he married an engineer named Hedvig Liebermann in Stockholm on August 13, 1947. They had two children named Ingrid and Lars. Shortly after they got married, they moved to the United States so Selberg could attend the Institute for Advanced Study in Princeton from 1947-48. He was offered another year but declined to look into other universities. However, in order to do this, he had to leave the country so he could obtain his new visa. So he and his wife travelled to Montreal. Paul Erdős arrived at the university when they got back. This was the time that the elementary proof of the prime number theorem was formed. Then in the following year, Selberg became an associate professor of mathematics at Syracuse University. He then returned to the Institute for Advanced Study at Princeton as a permanent member. A couple years later, Selberg was promoted to being a professor.

Before that, in 1950, Selberg was awarded the Fields Medal at the International Congress of Mathematicians at Harvard. The award was for his work on generalisations of sieve methods of Viggo Brun and his doctorate thesis back in 1943. This proof showed that a positive proportion of its zeros satisfy the Riemann hypothesis. In addition, he was also recognized for his elementary proof of the prime number theorem. Selberg wasn’t the first one to proving the prime number theorem nor did he bring up the problem in the first place. More of the history of the prime number theorem will be mentioned in another section. Selberg also go into a dispute with Erdős (which will be described more in-depth in another section). Selberg did not just provide an elementary proof of the prime number theorem; he did other mathematical work as well. He developed a number theory sieve where his method originated from the analytic theory of the Riemann zeta function (according to mathematician Enrico Bombieri), and introduced mollifiers (smooth functions with special properties). His work was published in two volumes - one in 1989 and the other in 1991.


In addition to the Fields Medal, Selberg also received the Wolf Foundation Prize in Mathematics with Samuel Eilenberg in 1986. This award was for “his profound and original work on number theory and on discrete groups and automorphic forms.” He also received the honorary award of the Abel Prize when it was established in 2002 for his status as one of the world’s leading mathematicians. Other honors included election into the Norwegian Academy of Sciences, Royal Danish Academy of Sciences, and American Academy of Arts and Sciences. In 1987, Selberg was named Commander with Star of the Royal Norwegian Order of St. Olav.

Selberg retired at the age of seventy in 1987. His wife died in July of 1995 and he remarried to a woman named Betty Frances Compton. At 90 years old, Selberg died of a heart attack. When he died, multiple tributes were paid to him for his outstanding mathematical work.
History of Primes

These kinds of numbers were first studied by the ancient Greeks. The mathematicians attending Pythagoras’s school were curious about their properties and tried to deduce patterns.

As time went on a mathematician named Euclid published a book (or rather books) called *Elements*. By the time this was published, which was around 300 BC, many discoveries about prime numbers have been made. For instance, Euclid proved in his 9th book that there are infinitely many prime numbers. This was proved using the method of contradiction. He also gave a proof of the Fundamental Theorem of Arithmetic which means that every integer can be written as a product of primes. He also proved that if $2^n - 1$ is prime, then the number $2^{n-1}(2^n - 1)$ is a perfect number. A little side note, Euler in 1747, proved that this can be shown for all even perfect numbers. No one knows for odd perfect numbers though.

Later in 200 BC, Eratosthenes, created an algorithm called the Sieve of Eratosthenes. This algorithm calculated primes.

Then no one made progress on this for numerous of years. This was known as the Dark Ages of the prime numbers.

Fast forward all the way to the 17th century. This is when Fermat came up with what he called Fermat’s Little Theorem. This theorem states that if $p$ is a prime number, then for any integer $a$

$$a^p = a \mod p$$

This theorem helped prove half of the Chinese hypothesis which dated back 2000 years earlier (the integer $n$ is prime if and only if the number $2^n - 2$ is divisible by $n$.

Fermat also communicated with other well-known mathematicians like Mersenne. He proposed that $2^n + 1$ is prime if $n$ is a power of 2. This was verified for 1, 2, 4, 8, and 16. However, when testing with the next case, it failed. $2^{37} + 1 = 4294967297$ and is divisible by 641.

Many mathematicians are familiar with Mersenne prime numbers which is written in the form $2^n - 1$. Although this is not true for all prime numbers, many mathematicians have been using this form to find the largest prime number. As of January 25, 2013, 48 Mersenne primes are known and the largest known one is $2^{57,885,161} - 1$.

There’s actually a website which posts the progress of Mersenne primes. It is located in the bibliography section of this paper.

Overall, many mathematicians have made a great impact when it comes to prime numbers (did not include Euler for example). From proof of the prime number theorem to discovering larger prime numbers, the concept can go a very long way.

The Dispute Between Selberg and Erdős

It all began in 1948. Selberg and Erdős were working together on developing an elementary proof of the prime number theorem that only used logarithms. Erdős announced they found the proof during this time. However, events led to an bitter argument between the two mathematicians. Dorian Goldfeld, a fellow colleague of both Selberg and Erdős contacted multiple sources in order to obtain the both of their works and compare them.

Before going into the work itself, we will look at the events that led to the dispute. In regards to Erdős’ paper, the Bulletin of the American Mathematical Society advised Erdős not to publicize his paper. He followed that advice and went on to the Proceedings of the National Academ of Sciences. However, Selberg went there as well. So both Erdős and Selberg’s papers were reviewed by a man named A.E. Ingham. Selberg received the
Fields Medal in 1950 and Erdős received the Cole Prize in 1952 partially because of their proofs of the prime number theorem. Now let’s look at the work that was done since 1948:

First we will define the prime number theorem by the following equation:

$$\theta(x) = \sum_{p \leq x} \log(p)$$

This equation denotes the sum of all primes $p \leq x$. This equation is equivalent to the assertion that

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1$$

Using this, in March 1948, Selberg proved that

$$\theta(x)\log(x) + \sum_{p \leq x} \log(p)\theta\left(\frac{x}{p}\right) = 2x\log(x) + O(x)$$

Selberg called this his fundamental formula. Erdős had no complaints and even complimented Selberg in the following quote from Erdős’s paper:

“Selberg proved some months ago the above asymptotic formula, ... the ingenious proof is completely elementary ... Thus it can be used as a starting point for elementary proofs of various theorems which previously seemed inaccessible by elementary methods.”

Selberg stated in regards to the formula “I found the fundamental formula ... in March this year [1948] ... I had found a more complicated formula with similar properties still earlier.”

A month later, April 1948, new conjectures were discovered. Below are definitions of two variables:

$$a = \lim \inf \frac{\theta(x)}{x}, \quad A = \lim \sup \frac{\theta(x)}{x}$$

(lim inf is defined as limit inferior, a limit that approaches a value from “below” and lim sup is limit superior, a limit that approaches something from “above”)

J.J. Sylvester, an English mathematician, gave a guaranteed estimate of $a$ and $A$ that

$$0.956 \leq a \leq A \leq 1.045$$

Selberg wrote a letter to H. Weyl on September 16, 1948 stating

“I got rather early the result that $a + A = 2$”

Goldfeld then proceeds to prove this estimate and generally verified that it was correct.

A little sidenote: Selberg was already aware of this estimate back in April. He also was aware that the prime number theorem would follow if one could prove $a = 1$ or $A = 1$. 

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Fast forward to May of 1948. According to a letter that Selberg wrote to Weyl in September 16, 1948, Selberg wrote down a sketch to the paper of Dirichlet’s theorem. In June, he took a break and prepared to go on a trip to Canada. In July, one of his fellow colleagues, Turán, asked him for his notes on the Dirichlet theorem because he also was leaving soon, as stated, “probably would have left when I [Selberg] returned from Canada.” Selberg gave him the notes and even sat down and went over it with him so Turán could understand the proof. He did mention his “fundamental formula” but did not give his proof nor his ideas about the formula. Using the notes, Turán gave a seminar of Selberg’s proof of the Dirichlet theorem. Erdős, along with other mathematicians like Chowla and Straus were there. Selberg did not mind this. He also did not mind that he mentioned the “fundamental formula” to Erdős.

In July of 1948, the dispute between Erdős and Selberg began. Erdős made the conjecture that one can derive the following from the Selberg’s inequality:

$$\lim_{n \to \infty} \frac{p_{n+1}}{p_n} = 1$$

($p_n$ is defined as the $n$th prime. So 2 is the 1st prime, 3 is the second prime, and so on.

After Erdős discovered this, Selberg made the following statement:

“...Actually, I didn’t like that somebody else started working on my unpublished results before I considered myself through with them.”

Selberg became paranoid of the possibility that Erdős might steal his work and take the credit for it. Because of this, Selberg worked the entire weekend right where Erdős left off. On Sunday, Selberg established his first proof of the prime number theorem. He said that he did not like it beause it was “long and indirect”. So after a few days, he developed a different proof.

Eventually, it was whoever came up with the best proof of the prime number theorem and how it should be published. The dispute was whether to accept only one mathematician’s entire proof or to collect each individual contribution and put Selberg and Erdős’ ideas all together. They both went off of the limit that Erdős produced and made multiple discoveries which produced a long proof on the prime number theorem.

On August 20, 1948, Selberg wrote a letter to Erdős with the following note. He generally stated that he did not like the fact that Erdős used his fundamental formula in order to prove the prime number theorem. Because of this, he said e was going to publish his proof as it was regardless of what happens. Selberg said that he will mention in the paper that his original proof depended on the result that Erdős got. Selberg also offered that he will withhold his proof so that Erdős proof can be published earlier without mentioning the prime number theorem. Basically, what Selberg is trying to mention here is that Erdős publishes his proof up to his recent discovery that he obtained using Selberg’s fundamental formulation.

Despite the fact that both Selberg and Erdős received awards for their proofs, they were still in dispute for many years. Up to 1997, each one still believes that their proof of the prime number theorem came first.
Preliminaries/Notations

Before we go in-depth with the proof of a portion of the prime number theorem, there are a few notes that need to be known before starting.

According to Selberg, the prime number theorem can be defined as:

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1$$

This is the equation he is trying to prove. The notation of $\theta(x)$ can be defined as:

$$\sum_{p \leq x} \log p$$

So as a quick example,

$$\theta(10) = \log 2 + \log 3 + \log 5 + \log 7$$

$$\theta(10.9) = \theta(10)$$ because they both have the same prime numbers less than or equal to the desired number.

There are other equations that define the prime number theorem. Here are two examples below:

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1$$

and

$$\lim_{x \to \infty} \frac{\pi(x)}{\int_{2}^{x} \frac{dt}{\log t}} = 1$$

These belong to Legendre and Gauss respectively.

$\pi(x)$ can be defined as the following:

$$\pi(x) = \text{the number of primes where } p \leq x = \sum_{p \leq x} 1$$

So $\pi(10) = 4$ since there are 4 prime numbers (2, 3, 5, 7) that are less than or equal to 10.

In the following portion of this proof, the notation of $R(x)$ will be used frequently and is defined as:

$$R(x) = \theta(x) - x$$
log \( x \) is defined a little differently than the usual log base 10. Instead, for this proof, log \( x \) is defined as the natural log of \( x \). So

\[
\log x = \ln x
\]

This paper also introduces the concept of “Big \( O \)”. of a function. \( (O(f(x))) \)

This symbol is known as Landau’s symbol as it was invented by German mathematician Edmund Landau.

This means that the function tends to infinity no faster than the function. This also means that constants are irrelevant in the equation. For instance:

Let

\[
f(x) = 4x^2 + 2x + 5
\]

Then applying our definition,

\[
f(x) = O(x^2)
\]

This means that \( f(x) \) increases no faster than a multiple of \( x^2 \)

One quick notation to touch base on for those who are not familiar with the notation is summation. The symbol for summation is below

\[
\sum
\]

Summation adds the function a certain amount of times until it reaches its limit. For example:

\[
\sum_{n=1}^{3} n = 1 + 2 + 3 = 6
\]

If the summation tends to infinity, the we take the limit of that summation. For instance:

\[
\sum_{n=1}^{\infty} 2^{-n} = 1
\]

because when we add all these numbers together \( (1/2 + 1/4 + 1/8...) \), the sum tends to 1.

Now in our case, we are dealing with summations that do not display an upper limit. Look back at one of the functions of the prime number theorem:

\[
\theta(x) = \sum_{p \leq x} \log p
\]

In this case the upper limit would be \( x \) and we are adding all the prime numbers less than or equal to \( x \) of the function. This is why \( \theta(10) = \log 2 + \log 3 + \log 5 + \log 7 \)
In the proof, \( p \) and \( q \) are variables for prime numbers and \( n \) is a variable for all integers.

\( x_0 \) is the lower bound of \( x \) contained in the subinterval \((x_0, x)\).

\( x \) is just a variable used that will help prove certain inequalities and equations.

\( K \) is defined as a constant variable.

**In-Depth Proof of a Portion of the Prime-Number Theorem**

We will be focusing on an inequality based off \( R(x) \) based off the existence of subintervals \((y, ye^{\delta/2})\) along \((x, xe^{\delta/2})\).

The in-depth analysis of this part of the proof is given below:

**To Prove:** The inequality Selberg proof in the second part of Selberg’s elementary proof of the prime number theorem, which proves basic equations, uses another inequality obtained earlier from in the fourth part of Selberg’s paper then gives the following:

\[
|R(x)| \leq \frac{1}{\log x} \sum_{n \leq x} |R\left(\frac{x}{n}\right)| + O\left(\frac{x}{\sqrt{\log x}}\right)
\]

\[
< K_4 \frac{x}{\log x} \sum_{(x/x_0)\leq n \leq x} \frac{1}{n} + \frac{x}{\log x} \sum_{n \leq (x/x_0)} \frac{1}{n} \frac{n}{x} R\left(\frac{x}{n}\right) + O\left(\frac{x}{\sqrt{\log x}}\right)
\]

Recall that in the second part of Selberg’s proof, he proved the basic formula that:

\[
|R(x)| \leq \frac{1}{\log x} \sum_{n \leq x} |R\left(\frac{x}{n}\right)| + O\left(\frac{x \log \log x}{\log x}\right)
\]

Then in the fourth part of the Selberg’s proof, he stated that for every \( x > 1 \):

\[
|R(x)| < K_4 x
\]

(*The notation of \( K_n \) is mentioned in the preliminaries of this paper*)

We need to look at how Selberg was able to put these two parts together to form the inequality that we want to prove. First, we need to prove how the first part the inequality is formed:

**Proof of the first inequality:** First begin with the following:

\[
2|R(x)| \log x \leq \sum_{p \leq x} \log p \left|R\left(\frac{x}{p}\right)\right| + \sum_{pq \leq x} \frac{\log p \log q}{\log(pq)} \left|R\left(\frac{x}{pq}\right)\right| + O(x \log \log x)
\]

This inequality was proven earlier by adding two inequalities together (which was also proven), and then obtaining the fact that \( O(x) \) is absorbed by \( O(x \log \log x) \).
It was also proved earlier that
\[
\sum_{p \leq x} \log p \left| R\left( \frac{x}{p} \right) \right| = \sum_{n \leq x} \left( \sum_{p \leq n} \log p \right) \left( \left| R\left( \frac{x}{n} \right) \right| - \left| R\left( \frac{x}{n+1} \right) \right| \right) + O(x)
\]

By substituting \( \sum_{p \leq x} \log p \left| R\left( \frac{x}{p} \right) \right| \) with its estimation, we can obtain the following:
\[
2 |R(x)| \log x \leq \sum_{n \leq x} \left( \sum_{p \leq n} \log p + \sum_{pq \leq x} \frac{\log \log q}{\log(pq)} \right) \left( \left| R\left( \frac{x}{n} \right) \right| - \left| R\left( \frac{x}{n+1} \right) \right| \right) + O(x \log \log x)
\]

Then, it was also proved earlier that
\[
\sum_{p \leq n} \log p + \sum_{pq \leq x} \frac{\log \log q}{\log(pq)} = 2n + O \left( \frac{n}{1 + \log n} \right)
\]

Now substituting for \( \sum_{p \leq n} \log p + \sum_{pq \leq x} \frac{\log \log q}{\log(pq)} \), we can obtain the following inequality:
\[
2 |R(x)| \log x \leq \sum_{n \leq x} 2n \left( \left| R\left( \frac{x}{n} \right) \right| - \left| R\left( \frac{x}{n+1} \right) \right| \right) + O \left( \sum_{n \leq x} \frac{n}{1 + \log n} \left( \left| R\left( \frac{x}{n} \right) \right| - \left| R\left( \frac{x}{n+1} \right) \right| \right) \right) + O(x \log \log x)
\]

As proved earlier
\[
\sum_{n \leq x} \frac{n}{1 + \log n} \left( \left| R\left( \frac{x}{n} \right) \right| - \left| R\left( \frac{x}{n+1} \right) \right| \right) = O(x \log \log x)
\]

and
\[
\sum_{n \leq x} 2n \left( \left| R\left( \frac{x}{n} \right) \right| - \left| R\left( \frac{x}{n+1} \right) \right| \right) = 2 \sum_{n \leq x} \left| R\left( \frac{x}{n} \right) \right| + O(x).
\]

Therefore, by substituting the appropriate parts, we obtain
\[
2 |R(x)| \log x \leq 2 \sum_{n \leq x} \left| R\left( \frac{x}{n} \right) \right| + O(x \log \log x)
\]

Then, divide both sides by \( 2 \log x \) to obtain the desired inequality. \( \text{(Note that } O\left( \frac{x \log \log x}{\log x} \right) = O\left( \frac{x}{\sqrt{\log x}} \right) \) \)

Now that we have proven the first part of the inequality, now we have to prove the second part of the inequality.
Proof of the second part of the inequality: If $x > x_0$, then

$$|R(x)| < K_4 \frac{x}{\log x} \sum_{n \leq x > x_0} \frac{1}{n} + \frac{x}{\log x} \sum_{x_0 \leq n \leq x} \frac{1}{n} \left| \frac{n}{x} R \left( \frac{x}{n} \right) \right| + O \left( \frac{x}{\sqrt{\log x}} \right)$$

Since we just proved the first part of the inequality, substitute that for $|R(x)|$.

Now we will split the sum that we have obtained:

$$\frac{1}{\log x} \sum_{n \leq x} \left| \frac{x}{n} \right| + O \left( \frac{x}{\sqrt{\log x}} \right) =$$

$$\frac{1}{\log x} \left( \sum_{n \leq x/x_0} \left| \frac{x}{n} \right| + \sum_{x/x_0 < n \leq x} \left| \frac{x}{n} \right| \right) + O \left( \frac{x}{\sqrt{\log x}} \right)$$

Now we can obtain the following equations:

Recall that

$$|R(x)| < K_4 x \text{ when } x > 1$$

Because of this, we can multiply by $x$ to obtain the following:

$$\frac{x}{\log x} \left( \sum_{n \leq x/x_0} \frac{1}{n} \left| \frac{n}{x} \right| R \left( \frac{x}{n} \right) \right) + \sum_{x/x_0 < n \leq x} \frac{1}{n} \left| \frac{n}{x} \right| R \left( \frac{x}{n} \right) + O \left( \frac{x}{\sqrt{\log x}} \right)$$

(We will be introducing $b$, which is $4\log 2 + 1$ and less than 8, to substitute)

$$\leq \frac{x}{\log x} \left( \sum_{n \leq x/x_0} \frac{1}{n} \left| \frac{n}{x} \right| R \left( \frac{x}{n} \right) \right) + \sum_{x/x_0 < n \leq x} \frac{b}{n} + O \left( \frac{x}{\sqrt{\log x}} \right)$$

Since $\sum_{x/x_0 \leq n \leq x} \frac{1}{n} = \log \left( \frac{x}{x/x_0} \right) = \log x - \log(x/x_0)$ by law of logarithms

then the above equation is equal to:

$$\frac{x}{\log x} \left( \sum_{n \leq x/x_0} \frac{1}{n} \left| \frac{n}{x} \right| R \left( \frac{x}{n} \right) \right) + b\left( \log x - \log(x/x_0) + O(\log x) \right) + O \left( \frac{x}{\sqrt{\log x}} \right)$$
Using basic algebra and the fact that \( O(x_0/x) = O(1) \), we can obtain

\[
\frac{x}{\log x} \left( \sum_{n \leq x/x_0} \frac{1}{n} \left| \frac{n}{x} R \left( \frac{x}{n} \right) \right| + b(\log x_0 + O(1)) \right) + O \left( \frac{x}{\log x} \right)
\]

\( b(\log x_0 + O(1)) \) tends to \( O \left( \frac{x}{\log x} \right) \). So substituting this, we get

\[
\frac{x}{\log x} \sum_{n \leq x/x_0} \frac{1}{n} \left| \frac{n}{x} R \left( \frac{x}{n} \right) \right| + O \left( \frac{x}{\log x} \right) + O \left( \frac{x}{\sqrt{\log x}} \right) =
\]

Since \( O \left( \frac{x}{\sqrt{\log x}} \right) \) produces a larger number than \( O \left( \frac{x}{\log x} \right) \), we can simply remove \( O \left( \frac{x}{\log x} \right) \) to obtain the following:

\[
\frac{x}{\log x} \sum_{n \leq x/x_0} \frac{1}{n} \left| \frac{n}{x} R \left( \frac{x}{n} \right) \right| + O \left( \frac{x}{\sqrt{\log x}} \right)
\]

Inserting this equation back into the inequality gives us the inequality as desired.

Now we combine the two inequalities that we have proven to get the longer inequality as desired.

\[ \text{QED} \]

**Remarks**

Prime numbers have been grabbing the interest of many mathematicians for numerous of years. They are astounded by discovering the next largest prime number or by deducing theorems on how each prime number is distributed. The art of prime numbers is a wonderful mystery to solve.

**References**

1. http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Selberg.html
7. http://kobotis.net/math/MathematicalWorlds/Fall2015/131/Projects/PNT/PNT.pdf