Liu Hui
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Liu Hui’s life

Not much is known of Liu Hui’s life. He was born in the beginning of the Three Kingdoms period and died at the end of it, a period considered one of the most romantic times in Chinese history, but how this affected his life is virtually unknown. His name, Liu Hui, is a Chinese name, which means that he was Hui of the Liu family and not the other way around. Western naming conventions would write it as Hui Liu.

It is possible that he was obscure during his lifetime, and that is why no record of his life was ever written; mathematicians were, although important, not considered an especially high position, and many Chinese mathematical works were anonymous.

But he does give one clue about his life in the preface to his commentary on the Nine Chapters of the Mathematical Art:

I read the Nine Chapters as a boy, and studied it in full detail when I was older, (Liu Hui n.d.).

From this, it is possible to assume that Liu Hui had been interested in mathematics from a young age.

He wasn’t of low birth; Liu Hui was a descendant of the Marquis of Zixiang, who ruled during the time of the Han Empire. This is likely the only reason he was able to access texts like the Nine Chapters on the Mathematical Art at all, let alone a young age as he claimed.

Nobles were the main producers of art, literature, and other advancements in the Three Kingdoms era. Particularly in Cao Wei, nobles were exempt from taxes and mandatory military service, which means that they were the only people who had the time to do things like create art and research the world around them.

Liu Hui also mentions something else of interest in the preface to his commentary:

"In the past, the tyrant Qin burnt written documents, which led to the destruction of classical knowledge...Because the ancient texts had deteriorated, Zhang Cang and his team produced a new version removing the poor parts and filling in the missing parts. Thus, they revised some parts, with the result that these were different from the old parts ..." ("Liu Hui” n.d.).
He was referring to the Qin dynasty, which preceded the Han dynasty. In 213 BC, the Qin ruler Shih Huang Ti attempted to reform learning by destroying all previous knowledge. Later, Zhang Cang did his restorations around 170 BC.

Liu Hui estimated the original Nine Chapters to be from around 1000 BC. However, it is possible that he was quite wrong about that; many historians believe it is actually from about 200 BC, after Shih Huang Ti’s burning of knowledge.

It is possible to take some more educated guesses about Liu Hui’s life from his work. He wrote very eloquently and quoted a number of works, which means one could assume that he was knowledgeable not only in mathematics, but literature as well. He seems like he might have been a very modest man - he never claimed results he was uncertain of, and instead preferred to leave it to coming generations of mathematician.

Let us leave the problem to whoever can tell the truth, (Liu Hui, n.d.).

He was shown to be someone who cared deeply about the economic status of his kingdom, as well as the suffering of the people. This, combined with his status as a noble, makes it possible to assume that he was a minor official in Cao Wei’s government.

Considering his comments in his commentary on the Nine Chapters, it is safe to assume Liu Hui never cared much for worrying about theoretical concepts. His main concern was how math could be used to solve real world problems more than anything else.

Liu Hui’s mathematical works

One of Liu Hui’s most well-known works is his commentary on The Nine Chapters on the Mathematical Art. The original text laid out procedures to solve various mathematical problems, but it gave no proofs it had instructions without explanation. In his commentary, Liu Hui did not write out proofs in the modern sense.

What Liu Hui did write were justifications for these procedures that could be used to write proofs in the modern sense. Instead of presenting the information and expecting anyone reading it to take its conclusions as granted, Liu Hui at least gave the principles on which the steps were based off of. Not only did he write justifications for the procedures in the text, but he also realized that many of them were approximations, and tested and refined them accordingly.

He was the first person in the world to propose the idea of decimals, and the first person to use them in his work.

Liu Hui dedicated himself to solving problems concerning area and volume, and in the process not only found solutions, but defined the terms themselves. He was particularly concerned with practical applications - very little of his work deals with theoretical concepts that had no immediate use in life at his time. He mentioned in the preface to his commentary on the Nine Chapters on the Mathematical Art that some of his discoveries would hopefully make the lives of peasants in his kingdom easier.

This was characteristic of ancient Chinese mathematicians in general; Chinese mathematics never delved very far into the theoretical.

One example of him refining approximations is when Liu Hui tested the Nine Chapters approximation of pi, which appears in the first chapter. He did this by applying a version of the Pythagorean Theorem, though he knew it as the Gougu Theorem; it had been independently discovered by Chinese mathematicians as well. Using this, he found a relation to express the length of the side of a polygon with $3 \cdot 2n$ in terms of the length...
of the side of a regular polygon with $3 \cdot 2^{n-1}$ sides.

The above diagram shows a circle with radius $r$ and center $O$. We already know that $AB = p_{n-1}$, which is the length of the sides of a regular polygon with $32^{n-1}$ sides. This means that $AY$ has length

$$\frac{p_{n-1}}{2}$$

Thus, $OY = \sqrt{r^2 - \left(\frac{pn - 1}{2}\right)^2}$

Then $YX$ has length

$$r - \sqrt{r^2 - \left(\frac{pn - 1}{2}\right)^2}$$

Since we now have $AY$ and $YX$, we can get $AX$ using the Pythagorean/Guogu theorem to get

$$\sqrt{r(2r - (4r - pn - 12))}$$

Then we know that $p_n = AX$ is the length of a side of a regular polygon with $N = 3 \times 2^n$ sides.

Liu Hui understood that we can get an approximation of $\pi$ as $\frac{N}{\pi} \frac{p_n}{2}$. This means that, as $N$ increases, the approximation becomes more and more precise. Using the perimeter of a hexagon, $6p_1 = 6$, gives an approximation of $\pi = 3$. Liu Hui obtained the approximation 3.14 by using a regular polygon with 96 sides, and then later obtained the even better approximation 3.14159 by using a regular polygon with 3072 sides. He did note that, for most practical purposes, 3.14 was sufficient.

Unlike Archimedes before him, Liu Hui didn’t use inscribed circles in his work, only circumscribed circles.
Some would say that, because of the many similarities between Euclid and Liu Hui's methods, that Liu Hui's work was influenced by the Greeks. This, however, was highly unlikely - there is no record of any direct interaction between a Chinese kingdom and any civilization that would have widespread knowledge of Ancient Greek mathematicians. The Silk Road only lasted from 206 BC to 220 AD, ending before Liu Hui was even born.

Not only that, but because of how radically different Chinese notation is from other notation systems used in Europe, it is much more likely that up until a certain point, Chinese advances in mathematics were in isolation up until long after Liu Hui had died.

However, it is true that Liu Hui had a similar influence on Chinese mathematics as Euclid had on Greek and European mathematics. He, ...established a conceptual framework to understand the logical interconnections of algorithms (Using History to Teach Mathematics: An International Perspective, 2000).

Liu Hui's process can be shown using a modern algebra program.

\[
n = 1, N = 6, p_n = 1, N/\frac{p_n}{n^2} = 3
\]

\[
n = 2, N = 12, p_n = 0.5176380900, N/\frac{p_n}{n^2} = 3.105828540
\]

\[
n = 3, N = 24, p_n = 0.2610523842, N/\frac{p_n}{n^2} = 3.132628610
\]

\[
n = 4, N = 48, p_n = 0.1308062584, N/\frac{p_n}{n^2} = 3.139350202
\]

\[
n = 5, N = 96, p_n = 0.06543816562, N/\frac{p_n}{n^2} = 3.141031950
\]

\[
n = 6, N = 192, p_n = 0.03272346325, N/\frac{p_n}{n^2} = 3.141452472
\]

\[
n = 7, N = 384, p_n = 0.01636227920, N/\frac{p_n}{n^2} = 3.141557606
\]

\[
n = 8, N = 768, p_n = 0.008181208047, N/\frac{p_n}{n^2} = 3.141583890
\]

\[
n = 9, N = 1536, p_n = .004090612582, N/\frac{p_n}{n^2} = 3.141590463
\]

\[
n = 10, N = 3072, p_n = 0.002045307359, N/\frac{p_n}{n^2} = 3.141592104
\]

(Liu Hui n.d.)
Of course, Liu Hui did not use the same algebraic notation as above, nor did the same number system, but this still shows that he understood the concept of a limit, even if it was an unpolished understanding. Through all of this, he was able to show that,

... multiplying half the diameter and half the circumference, one gets the area. (Liu Hui n.d.).

Essentially, he independently developed the method of exhaustion and used it in his approximations. Unlike Euclid and Archimedes before him, however, he had the advantage of the concept of zero, as well as the decimal system that he introduced, to help him in his work.

It is clear just by looking at his process that he understood the idea of a limit - he couldn't directly test a circle, so he just used shapes with many sides that came closer and closer to what a pi actually is. He knew he was unable to get the exact approximation, but he came much closer than anyone else had by that time.

As he said,

The smaller they are halved, the finer [Xi] are the remaining [dimensions]. The extreme of fineness is called subtlety [Wei]. That which is subtle without form [Xing]. (Qian 1968, translated by Wagner 1979).

However, Liu Hui did not feel too disappointed that he could find the exact value of pi. He was only concerned with real world purposes and paid little attention to theoretical matters, as shown in this quote:

When it is explained in this way, why concern oneself with the remainder? (Qian 1968, translated by Wagner 1979)

3.14 was all one needed for practical applications. To Liu Hui, mathematics was simply a means to an end.

But finding an approximation of pi is not his only achievement. Another one of his contributions is in his commentary on the fifth chapter in the Nine Chapters on the Mathematical Art. In it, he is able to calculate the volume of various shapes, including pyramids, prisms, cones, tetrahedrons, wedges, cylinders, and the frustums of cones.

However, one shape eluded him. Liu Hui was unable to find the area of a sphere, and said that it would be up
to future mathematicians to succeed where he had failed.

In chapter eight, Liu Hui looks at simultaneous linear functions, and uses both positive and negative numbers in his computations.

But his commentary on the Nine Chapters on the Mathematical Art is not Liu Hui's only work. He also wrote something called the Sea Island Mathematical Manual. This manual contains 9 problems, and it was originally part of his commentary on the Nine Chapters, but later on it was removed and made into its own separate work.

It shows how to use the Gougu/Pythagorean theorem to calculate the heights of objects, and distance to objects, in cases where a direct measurement is impossible. This has many applications, and the first question details the style in how it is done. It concerns an island in the sea, and is the source of the name of the work.

Poles $p_1$ and $p_2$ are both 5pu* high and 1000pu apart. $x$ is 123pu from $p_1$ and $y$ is 127pu from $p_2$, at ground level.

When viewed from $x$, the summit $s$ is in line with the top of $p_1$ and when viewed from $y$, the summit $s$ is in line with the top of $p_2$.

Liu Hui's formula was that, if the height of the poles is $h$ and the distance between the poles is $d$, then the height of the island can be found using the formula

$$ h \times \frac{d}{p_2y - p_1x} + h $$

And the distance to the island

$$ p_1x \times \frac{d}{p_2y - p_1x} $$

In this particular example problem, Liu Hui gave the height of the island as 1255pu and the distance to the island 30750pu.

*Pace units, or pu, were an outdated unit of measurement from China. 1pu was equal to roughly 2m.

There were many similar problems in the book, including the distance to a square town, the depth of a gorge, the the depth of a valley with a lake at the bottom, the size of a town seen from a mountain, the height of a tree on the side of a mountain, the height of a tower on a hill, the width of a river, the width of a ford viewed from a hill, and many more.

He had many accomplishments in number theory as well. Liu Hui was one of the first people to introduce the idea of simplifying fractions and finding the common denominator so the fractions can be added. He also
discussed the irrationality of many square roots, and suggested that a 10-based system - what we know today as decimals - could be used to more easily express such things.

Additionally, he codified ratios—called iv in Chinese— which helped him to develop the idea of matrices.

Collaboration with other scholars

It is unknown if Liu Hui collaborated with any of his contemporaries. He did not mention doing so, thus it is reasonable to assume that he did not.

However, his work was definitely built on the achievements of his predecessors. His most famous work is his commentary on The Nine Chapters on the Mathematical Art.

Liu Hui wrote that he believed the Nine Chapters to be a very old document; his estimate placed its origin at around 1000 BC. However, modern historians are of the opinion that it originated from 200 BC.

He credited Zhang Cang with restoring and revising the original document in 170 BC. However, some historians disagree with him and say it couldn’t be from earlier than 200 BC.

The Nine Chapters uses measurement notation that originated from 200 BC, but not later notation, which would lead one to conclude that it originated from shortly after 200 BC. At the same time, the Nine Chapters were an important document, and it is possible the measurements would have been updated as time passed.

Either way, Liu Hui’s accomplishments would have been impossible, or at the least lesser, without the originalNine Chapters on the Mathematical Art.

Historical events that marked Liu Hui’s life.

Liu Hui was born in one of the most interesting as well as the bloodiest periods in not just Chinese history, but the history of the world.

The Three Kingdoms era was a time of both great intrigue and great tragedy; a census recorded at the end of the previous Han dynasty showed the population of China to be around 56 million people, but by the end of the Three Kingdoms era, the population of all of China had fallen to 16 million (History of Three Kingdoms Period n.d.).

The Han Empire—the empire preceding the Three Kingdoms—lasted from 206 BCE-221 CE, but its decay began decades before it finally collapsed. By the 180s CE, the government had grown corrupt and the courts unstable. Army generals, scholar officials, and court eunuchs all vied for power, which culminated in the assassination of Emperor Shao, who was only 13.

The Han Empire was struggling financially as well; the scholar officials exempted themselves from taxes, and as the population grew, peasants found it very difficult to support themselves and their families at all—let alone pay enough taxes to keep the empire afloat.

In 220, Cao Pi proclaimed himself emperor, with Wei as his national title. In 221, Liu Bei proclaimed himself emperor as well, with Shu as his national title. And then, in 229, Sun Quan claimed himself to be emperor too, with Wu being his national title. This created the titular three kingdoms—the Wei Empire, the Shu Empire, and the Wu Empire. Liu Hui was born in Wei (also known as Cao Wei, to distinguish it from other Wei Empires) in 225.
Cao Wei's first emperor, Cao Cao, made it a priority to reform the government instead of keeping old policies left over from the Han Empire. He implemented a system where all of the men of certain families would be required to either fight or do menial labor, meaning Cao Wei had a steady supply of soldiers. Cao Cao also reformed the tax system in a way that put less burden on the peasant farmers, as well as adopted policies to improve irrigation and farmland.

All of these factors led to Cao Wei arguably having the most internal stability of the kingdoms.

None of the kingdoms were able to advance economically much, as the constant chaos of warfare strained their resources. However, many great poets and artists lived in this time, and the use of clay puppets to tell stories and act out plays originated here. The Three Kingdoms era became very popular as the subject of numerous poems and plays, the most famous and influential being Romance of the Three Kingdoms.

Significant historical events around the world during Liu Hui’s life

China was not the only place undergoing upheaval in the 3rd century. The Roman Empire at this time was at its peak in size, covering almost all of western and southern Europe, as well as large parts of the Middle East and North Africa. However, the larger the empire became, the harder it was to maintain, and thus the Roman Empire became more and more unstable. This was called the Crisis of the Third Century.

In 200 AD, Septimus Severus challenged Didius Julianus, who at the time was the highest candidate for next emperor, at an auction, and one, taking the throne. He then defeated Septimius Albinus and Pescennius Niger, two other contenders, and became emperor himself.

However, twelve years later, his own son Caracella took the throne and killed his own brother. He also issued the Caracella Edict, which gave citizenship to every free person living in the empire. The problem with this was that citizens had few rights, so it did not make much of a difference.

During the Crisis of the Third Century, a part of the Roman Empire broke away and became the Palmyrene Empire. The Palmyrene Empire encompassed large parts of the Middle East. However, the empire was very short lived; it lasted from 270-273 before it was reconquered by the Romans. Its legacy today is as a symbol of Syrian nationalism.

China was also not the only region split into three powers. The Korean peninsula was also split into the three kingdoms of Paekje, Silla, and Koguryo (the last of which the modern name "Korea" originated).

Although all three of these kingdoms existed since around the first century BC, it was not until 200 AD that they all had definite borders and rulers. The first Korean writing system was created in this time, using a combination of Chinese letters and phonetic characters.

In 210 AD, Koguryo fought with forces from Kung-sun, a tribe from eastern China, and was defeated in their fortress along the T’ung-chia River. Thus they moved south to build a new city, Wan-tu, in the T’ung-chia plain, and many burial mounds and paintings have been found there.

Later, in 238 AD, Cao Wei invaded the Kung-sun, and so the kingdom of Koguryo made an alliance with Cao Wei, forcing the Kung-sun to fight on two fronts - both the north and the east. Koguryo also forged relations with Dong Wu and sent emissaries to make alliances with distant kingdoms, such as the Sun Ch’iian.

However, after the Kung-sun were eliminated by the double assault in 238 AD, Cao Wei and Dong Wu turned on Koguryo and attempted to invade it.

In 244 AD, General Wu Ch’iu Chien managed to capture the Koguryo city of Wan-tu, and forced Koguryo
even farther east, into what is now Hamgyong-do. In 250 AD, the Kofun period of Japan began. The Kofun period is the earliest recorded period in Japanese history, and the name "Kofun" means "mounds", referring to the elaborate burial mounds that originated from this time.

Meanwhile in Persia, the Shah Ardashir defeated the Parthian Empire battle of Hormizdagan. This was the start of the Sassanid Empire.

The Sassinids waged many successful campaigns against the Romans throughout the 200s, and even managed to take the emperor Valerian as a prisoner. Many reforms were made in this era, in an attempt to eliminate Greek influence. Between 260 and 263 AD, many reforms in the tax system were made, as well as careful urban planning. Zoroastrianism became the state religion during this time, and priests held much power in society.

Developments continued in the Americas, too. Around 250 AD, the Mayan preclassical period ended and the classical period began, and many of their greatest innovations are estimated to have been made around this time. While Liu Hui was writing his works and solving his problems, so too were the Mayans solving mathematical problems relevant to them at the time.

Many Mayan advances were guided by their religion. They believed in the significance of celestial bodies in everyday life, which encouraged them to develop a calendar almost as precise as the one we use today. It also led to their independent discovery of the concept of zero, which is just another example of how mathematics transcends all cultural boundaries.

Near the Mayans, in what is now modern day Mexico, was Teotihuacan. It is unknown if this was simply a city-state, or the center of an empire, but either way, the city of Teotihuacan was one of the greatest cities of the 3rd century. It was the 6th largest city in the world, after Constantinople, Changan, Loyang, Ctesiphon, and Alexandria.

The city was extremely well planned, and though they left few written records, their architecture was truly incredible. It was believed that the people of Teotihuacan traded with Mayans to the south, and that they shared mathematical achievements. The two peoples had similar religious beliefs, and many buildings in Teotihuacan suggested that they had a strong understanding of geometry and how it could be applied to architecture.

**Significant mathematical progress during Liu Hui’s lifetime**

The evolution of China’s measuring units was integral to the creation of the power. The first measuring system was based on body parts - a man’s hand, a woman’s hand, a forearm, a foot, etc. However, during Shih Huang Ti’s reign, as part of his desire to reform education, he chose the number 6 as his emblem and had only 6 main measuring units.

However, using only 6 units of measurement overall would be incredibly impractical. Thus, the power was developed.

For example, 1 chihh tshun would equal 10 chihh, 1 chihh fen would equal 100 chihh, 1 chihh li would equal 1000 chih, and so on.

The powers could be distinguished from numbers by their placement. The characters for various powers were placed in the upper part of the character, thus the symbol for tshun could be distinguished from the character for 2 because of its slightly higher and smaller position. This is similar to how modern exponent notation makes the exponent smaller and places it near the top of the number.
Chinese people did have the concept of negative numbers long before Europe even considered them. In Liu Hui's work, they were represented by normal rod numerals slanted slightly. However, negative exponents were not introduced until much later.

They never concerned themselves with the square roots of negative numbers. Though, the Indians were at least aware of imaginary numbers and complex numbers, but did not delve into this very much at the time.

**Connections between history and the development of mathematics**

The Three Kingdoms period, which spanned Liu Hui's entire life, was a very bloody time with many battles and much bloodshed. However, as stated previously, the lack of information on Liu Hui's life means it is uncertain how the warfare affected him personally. None of his work seemed to have immediate military applications.

However, Liu Hui could have only found the ability and time to develop his theories because he was a minor official in Cao Wei. Being exempt from taxes and military service meant he wasn’t conscripted to die in a war against Dong Wu, Shu Han, the Kung-sun, Koguryo, or any of the other forces Cao Wei fought with.

This was also true for many other scholars of this period; the free time and money provided by their position meant they could focus on intellectual pursuits. At the same time, it is unfortunate to think about how many brilliant people happened to be peasants, and died in combat instead of making their own discoveries.

**Remarks**

Liu Hui definitely deserves his position as one of the great mathematicians of Chinese history, as well as one of the greatest in the entire world.

**References**
