Joseph Liouville was the first to discover transcendental numbers. His mathematical works ranged from mathematical physics, astronomy, and pure mathematics. A little about Liouville is that he attended a prestigious university called Ecole Polytechnique studying rigorous mathematics. Although he did not attend any of Cauchy's lecture, it is clear that Cauchy was a major influence on Liouville. He has accomplished a lot prior to settling with an academic career. He has written papers to the Academu on electrodynamics, partial differential equations, and the theory of heat. In 1836, Liouville published a mathematical journal called *Journal de Mathématiques Pures et Appliquées*, which contributed to the mathematics in France. He was promoted to different positions over the course of two years. He had many responsibilities that he split by the seasons. When he began his work as a politician, his mathematical output declined. Joseph Liouville is remembered today for constructing an infinite class of transcendent numbers using continued fractions. He specified number .1100010000000000000000000000000001000... and it is called the Liouville Number. It is said that his idea stemmed from reading a correspondence between Goldbach and Bernoulli. His first goal was to prove that $e$ is transcendental, but failed. After continuous work and dedication, he finally proved the existence of a transcendental number. He discovered this when he constructed an infinite class of such numbers using rational numbers or fractions. Liouville published his results on transcendental numbers in 1851. Besides Joseph Liouville mathematical contributions, he has some interesting facts. For example, Joseph Liouville escaped from France during the 1848 revolution because he was avoiding a prison sentence for stealing precious books and manuscripts. Even though Liouville has health problems, he accomplished his working during these years of his life. Liouville wrote about 200 papers in number theory, and in total he wrote over 400 papers in total. That me be a large amount of papers, but still not more than Leonhard Euler.

**Definition of Liouville Numbers**

"Liouville numbers are irrational numbers $x$ with the property that, for every positive integer $n$,
there exist integers p and q with q > 1 such that
\[ 0 < \left| x - \frac{p}{k} \right| < \frac{1}{q^n} \]

" (Wikipedia) Liouville numbers are a form of transcendental number, which is a real or complex number that does not satisfy a non-zero polynomial equation with integer coefficients. The Liouville number was the initiation of transcendental numbers in 1844.

**Liouville Number Proof**

*Proof Provided by Source (4)*

In order to understand what Liouville numbers there must be some terms that are defined and explained such as mean value theorem, algebraic numbers, and transcendental numbers. Transcendental numbers is defined as a real or complex number that is not a solution of a non-zero polynomial equation. Some transcendental numbers are can be described with continuous functions or as a limit of an infinite series. For example, \( \pi \) can be described with the fraction \( \frac{355}{113} \), which express \( \pi \) accurately by six digits.

Algebraic Numbers: An algebraic number is any complex number is a root of a non-zero polynomial in one variables such as \( x \), with rational coefficients (or equivalently by clearing denominators with integer coefficients). All integers and rational numbers are algebraic, as are all roots of integers. For example, \( 2x^2 - 4x + 2 \), this equation can be solved to find that \( x \) is an algebraic number. A transcendental number cannot fall under these guidelines.

Liouville number is a transcendental numbers, and is the simplest transcendental numbers that can be justified. The Liouvillian Number is the summation of \( 10^n \). The number is written as \( .1100010000000000000000010000.. \) where there is a 1 in place \( n! \) and 0 everywhere else. Transcendental numbers cannot be algebraic numbers because the can be a selection i the prison.

Mean Value Theorem is defined states that if \( f(x) \) is defined and continuous on the interval \([a,b]\) and differentiable on \((a,b)\), then there is at least one number \( c \) in the interval \((a,b)\), simply meaning that \( a<c<b \) such that
\[
f'(c) = \frac{f(b) - f(a)}{b-a}
\]

All liouville numbers are transcendental.

Let \( \alpha \) be an irrational number which is a root of \( f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x] \) with \( f(x) \not= 0 \). Then there is a constant \( A=A(\alpha) >0 \) such that if \( a \) and \( b \) are integers with \( b>0 \), then
\[
| \alpha - \frac{a}{b} | > \frac{A}{b^n}
\]

. The lemma for liouville numbers begins: Let \( M \) be the maximum of \( | f'(x) | \) on \([\alpha-1,\alpha+1]\]. Let \( \alpha_1, \alpha_2, ..., \alpha_m \) be the distinct roots of \( f(x) \) which are different from \( \alpha \). Fix
\[
A<\min\{1,1/M, | \alpha - \alpha_1 |, | \alpha - \alpha_2 |, ..., | \alpha - \alpha_m | \}
\]

Assume (1) does not hold for some \( a \) and \( b \) integers with \( b > 0 \). Then
\[
| \alpha - \frac{a}{b} | \leq \frac{A}{b^n} \leq A<\min\{1, | \alpha - \alpha_1 |, | \alpha - \alpha_2 |, ..., | \alpha - \alpha_m | \}
\]
Hence, \( \frac{a}{b} \notin [\alpha - 1, \alpha + 1] \) and \( \frac{a}{b} \notin \{\alpha_1, \ldots, \alpha_m\} \).

By the Mean Value Theorem, there is an \( (x_0) \) between \( \frac{a}{b} \) and \( \alpha \) such that
\[
f(\alpha) - f(\frac{a}{b}) = (\alpha - \frac{a}{b})f'(x_0)
\]
so that
\[
| \alpha - \frac{a}{b} | = \left| \frac{f(\alpha) - f(\frac{a}{b})}{f'(x_0)} \right| = \frac{| f(\frac{a}{b}) |}{| f'(x_0) |}
\]
Since \( f(\frac{a}{b}) \neq \), we deduce that
\[
| f(\frac{a}{b}) | = \left| \sum_{j=0}^{n} a_j b^{n-j} \right| / b^n \geq 1 / b^n
\]
Thus, since \( | f'(x_0) | \leq M \), we obtain
\[
| \alpha - \frac{a}{b} | \geq \frac{1}{Mb^n} > A \geq | \alpha - \frac{a}{b} |
\]
giving a contradiction. Earlier in the text, it states the definition of a Liouville number. Let \( \alpha \) be a Liouville Number. First, we show that \( \alpha \) must be irrational. Assume \( \alpha = \frac{c}{d} \) for some integers \( c \) and \( d \) with \( d > 0 \). Let \( n \) be a positive integer with \( 2^n > d \). Then for any integers \( a \) and \( b > 1 \) and \( a/b \neq c/d \), we have that
\[
| \alpha - \frac{a}{b} | = \left| \frac{c}{d} - \frac{a}{b} \right| \geq \frac{1}{bd} < \frac{1}{2^{n-1}b} \geq \frac{1}{b^n}
\]
This follows the definition of a Liouville number that \( \alpha \) is not a Liouville number giving a contradiction. Therefore, means that \( \alpha \) is irrational. Assume that \( \alpha \) is an irrational algebraic number. By the lemma, there exists a real number \( A > 0 \) and a positive integer \( n \) such that (1) holds for all integers \( a \) and \( b \) with \( b > 0 \). Let \( r \) be a positive integer for which \( 2^r \geq 1 / A \). Since \( \alpha \) is a Liouville Number, there are integers \( a \) and \( b \) with \( b > 1 \) such that
\[
| \alpha - \frac{a}{b} | \leq \frac{1}{b^n} < \frac{1}{2^r b^n} \leq \frac{A}{b^n}.
\]
This contradicts (1) and, hence, establishes \( \alpha \) is transcendental. An example to show \( \alpha = \sum_{j=0}^{n} \frac{1}{2^j} \) is a Liouville number. The fact that the binary expansion of \( \alpha \) has a long string’s of zeros, therefore, means that it cannot be rational. Fix a positive integer \( n \) and consider \( \frac{a}{b} = \sum_{j=0}^{n-1} \frac{1}{2^j} \) with \( a \) and \( b = 2^n > 1 \) integers. Then
\[
0 < | \alpha - \frac{a}{b} | = \sum_{j=n}^{\infty} 2^{-j} \leq \sum_{j=n}^{\infty} 2^{-(n+1)} = 12^{-n+1} \leq \frac{1}{2^n} = \frac{1}{b^n}
\]
The Liouvillian Number is a number that is described as \( \sum_{j=n}^{\infty} 10^{-j} \). The proof describes that transcendental numbers are irrational and are not solutions to rational coefficient polynomials. These numbers are compiled with continuous rational fractions. These numbers cannot be algebraic numbers. The incorporation of the mean value theorem demonstrates that there is a transcendental number that lies between two intervals such as (0,1). The proof shows that \( x \) is irrational, and it shows that the difference between \( x \) and any \( p/q \), which explains the 0 < part.
Other uses of Liouville Number and Transcendental Numbers

Transcendental numbers are used to decipher the difference amongst real sets of numbers. There are transcendental numbers such as pi and e that are used to calculate many things such as area, circumference, and etc. Pi is the world renowned because it used in many advanced civilizations to determined the circumference and area of a circle. E is used in many famous equations such as Euler’s mathematical constant $e^n = -1$ Another equation that incorporates e is

Remarks

Throughout my academic career, I have enjoyed the subject mathematics because it was the one subject that I understood. Other subjects like reading, english, and science don’t always have a definite answer. It usually comes easy for me. When I entered Professor Kobotis class I was overwhelmed with the new addition of mathematics, I really was not sure how to retain all the information. I was introduced to many new fields of mathematics, as well as the history behind many of these fields. In today’s society, it has been different debates about the removal of calculus and different maths in curriculums. However, it is as necessary as other aspects that are taught like poetry in english class, and science. It would be a travesty if mathematics was taken away because it is another way that stimulates the mind and sparks new ideas. I have learned a lot especially through the different papers that were assigned throughout the course of the year.

This research about Liouville numbers allowed me to learn many things. When I first heard of transcendental numbers, I assumed they were not real. It is interesting to discover that Algebraic numbers are countable, while transcendental numbers were uncountable. Many would think that algebraic numbers consisted of just more numbers, but that is not true. The argument is describes most real numbers as transcendental, and because real numbers are uncountable that means transcendental numbers are also uncountable. This means there are more transcendental numbers than algebraic numbers. A transcendental function follows the same rules as a transcendental number, like the function sine(x).

References