The Uncountability of Transcendental Numbers
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The history of transcendental number The name "transcendental" comes from the root trans meaning across and length of numbers and Leibniz in his 1682 paper where he proved that sin(x) is not an algebraic function of x. Euler was probably the first person to define transcendental numbers in the modern sense.

After the Launielle number proved, many mathematicians devoted themselves to the study of transcendental numbers. In 1873, the French mathematician Hermite (Charles Hermite, 1822-1901) proved the transcendence of natural logarithm, so that people know more about the number of transcendence. In 1882, the German mathematician Lindemann proved that pi is also a transcendental number (completely deny the "round for the party" mapping possibilities). In the process of studying transcendental numbers, David Hilbert has conjectured that a is an algebraic number that is not equal to 0 and 1, and b is an irrational algebra, then $a^b$ is the number of transcendences (Hilbert’s problem of the seventh question). This conjecture has been proved, so we can conclude that e, pi is the transcendental number.

$\pi$ The first to get $\pi = 3.14 \ldots$ is the Greek Archimedes (about 240 BC), the first to give $\pi$ decimal after the four accurate value is the Greek Ptolemy (about 150 BC), the earliest Calculate the $\pi$ decimal after seven accurate value is Chinese ancestor Chong (about 480 years), in 1610 the Dutch mathematician Rudolph application of internal and external tangent polygon calculation of $\pi$ value, through the calculation of $\pi$ to 35 decimal digits. He spent a lifetime of energy, in 1630 Greenberg improved Snell’s method to calculate the $\pi$ value to 39 decimal places, which is the use of classical methods to calculate the $\pi$ value of the most important attempt.

It is worth mentioning that Dash was born in Hamburg in 1824, lived only a short period of 37 years, then passed away. He is a lightning calculators, is one of the greatest artificial calculators, he was 54 seconds The clock completes two 8-bit multiplications, completes two 20-bit multiplications in six minutes, and completes two 40-bit multiplications in 40 minutes; he calculates a 100 in 52 minutes The square root of the number of digits. Dash ‘s extraordinary calculation can be in his production of $7 - \log$ and from 7,000,000 to 10,000,000 between the number of factor table will be the most valuable full use. In 1706 William Smith in England first used the notation $\pi$ to denote the ratio of the circumference to the diameter, but only after Euler had adopted this method in 1737, $\pi$ was used.
In 1873, the British Williamson uses the new wheat formula to calculate the 
pi to 70 bits. In 1961, the United States Leisiqi and D Sankesi obtained with the electronic computer 
? value of 100000 digits.

\textbf{e} In 1844, the French mathematicians speculated that e is beyond the number, until 1873 only by the 
French mathematician Hermite proved that e is beyond number.

In 1727, Euler first used e as a mathematical symbol use, and later after a period of people also deter-
mined to use e as the natural logarithm of the end to commemorate him. Interestingly, e is the name 
of the first lower case Euler, is intentional or accidental coincidence? We can’t figure out now.

\textit{e} in the application of natural science and no less than pi value. Such as atomic physics and geology 
to study the decay laws of radioactive material or study the Earth’s age will be used to e.

\textit{e} will be used to calculate the rocket speed using the Zolkovsky formula. \textit{e} should also be used when 
calculating the optimal interest rate for savings and the problem of biological reproduction. The same 
as \?, \textit{e} will also occur in unexpected places, such as: ”to be a number divided into several equal parts, 
to make the aliquot of the product, how?” To solve this problem will have to deal with \textit{e}. The answer 
is: Make the aliquots as close to the \textit{e} value as possible. For example, the 10 is divided into 10/\textit{e} = 3.7 
copies, but 3.7 copies of bad points, so divided into 4 copies, each 104 = 2.5, 2.54 = 39.0625 product, 
such as divided into 3 or 5 copies, The product is less than 39. \textit{e} is such a magical appearance.

In 1792, the 15-year-old Gauss discovered the prime number theorem: ”The percentage of prime num-
bers from 1 to any natural number N is approximately equal to the reciprocal of the natural logarithm 
of N; the bigger the N, the more accurate the law.” In 1896 by the French mathematician Adama 
and almost the same period of the Belgian mathematician cloth scattered proof. To \textit{e} for the end 
there are many advantages. Such as to \textit{e} for the end of the preparation of the best logarithmic table; 
calculus formula also has the most simple form. This is because only the \textit{e} derivative is itself, that is 
\(\frac{d}{dx}(e^x) = e^x\).

\textbf{Georg Cantor}

Georg Ferdinand Ludwig Philipp Cantor (March 3, 1845 - January 6, 1918), a German mathematician 
(Baltic German) born in Russia. He founded the modern set theory as a real number theory as well 
as the whole calculus theory system foundation. He also proposed the conception of potential and 
order of sets. However, his research results were not recognised, led by Leopold Kronecany and other
mathematicians, they started long-term attacks toward Cantor. Later on, he suffered from depression and mental disorders. Since 1869 he taught at the University of Halle. And he died in 1918. For more than two thousand years, scientists have been exposed to the infinite, but unable to grasp and understand it. This is indeed a tough challenge to mankind. Cantor’s unique thought, his rich imagination, a novel method of drawing a human wisdom of fine - set theory and the theory of super-poor number. What he figured out made the entire math world or even philosophy field feel shocked in 19, 20 th century. It is no exaggeration to say that ”the revolution on mathematics is almost done by this independent man.

**Set Theory** In the 19th century, due to the rigor of analysis and the development of function theory, mathematicians put forward a series of important problems and made a serious study of irrational number theory and discontinuous function theory. This research laid the foundation for Cantor’s later work. The necessary ideological basis.

While Cantor was finding the function of the triangular series expansion of the uniqueness of the criterion of discrimination, he realized the importance of the infinite sets, and began to engage in the infinite set of general theoretical research twice. As early as 1870 and 1871, Cantor published in the ”Journal of Mathematics” papers, proved the function f (x) of the trigonometric series representation of the uniqueness theorem, and proved that even in a limited number of discontinuities Convergence, the theorem is still established. In 1872 he published an article entitled ”The Generalization of a Theorem in the Trigonometric Series” in the Annals of Mathematics, extending the uniqueness of the results
to allow an exceptional value to be an infinite set of situations. In order to describe this set, he first defines the limit points of the point set, and then introduces some important concepts such as the set of the point set and the set of the derivative. This is the beginning of the research on the problem of uniqueness and the theory of point set theory. Later, he in the "Yearbook of Mathematics" and "Journal of Mathematics" published two articles on many articles. He called the collection for a certain, different things in the overall, these things people can realize and can determine whether a given thing belongs to this population. He also pointed out that if a collection can and its part of a one-to-one correspondence, it is infinite. He also gives the open set, closed set and complete set of other important concepts, and defines the set and the intersection of two operations.

In order to generalize the concept of the number of elements of finite sets to infinite sets, he proposed the concept of set equivalence with the principle of one-to-one correspondence. Only two sets of their elements can be established between the one-to-one correspondence is called equivalent. This is the first time on a variety of infinite set by their elements of the "number" of the classification. He also introduced the concept of "can be listed", to any positive and can constitute a one-to-one correspondence is called a countable set. In 1874 he published a paper in the Journal of Mathematics that proved that the set of rational numbers is listed, and later he also proved that all the algebraic number of the overall composition of the set is also available. As to whether the set of real numbers can be listed, Cantor gave a letter to Dedekind in 1873, but soon he got the answer: the real number set is not listed. Since the set of real numbers is not columnable, and the set of algebraic numbers is permissible. He obtained the conclusion that there must be transcendental numbers, and the transcendental number is "much more than" algebraic numbers. In the same year, we construct the famous Cantor set of the real function theory, and give an example of the uncountable set with zero measure. He also skillfully corresponds to a point on a straight line and a point on the whole plane, and even a straight line with the entire n-dimensional space for a one-to-one correspondence. From 1879 to 1883, Cantor wrote six series of papers, the total topic is "On the infinite linear point manifold", of which the first four with the previous paper is similar to the discussion of the set theory of some of the mathematical results, in particular Is concerned with some interesting applications of set theory in the analysis. The fifth paper was published in a single book, the book's title "general set theory basis." The sixth paper is the fifth supplement. Cantor’s doctrine is: "Mathematics in its own development is completely free, the concept of his only restriction is: must be non-contradictory, and with the introduction of the concept by the precise definition of coordination ... ... the nature of mathematics It is in its freedom.

**Something about transcendental number**    The existence of the transcendental number was first
proved by the French mathematician Joseph Liouville (1809-1882) in 1844. On the existence of transcen-
dence, Joseph wrote the following infinite decimal: \( a = 0.11000100000000000000001000 \ldots \), and
prove that taking this a cannot satisfy any integer coefficient algebraic equation, thus proving that it is
not an algebraic number, but a transcendence number. Later, in order to commemorate his first proof
of transcendence, so the number a called Liouville number.

After Joseph proved the assumption, many mathematicians devoted themselves to the study of tran-
scendental numbers. In 1873, the French mathematician Hermite (Charles Hermite, 1822-1901) proved
the transcendence of natural logarithm e, so that people know more about the concept behind transcen-
dental numbers. In 1882, the German mathematician Lindemann proved that pi is also a transcendental
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the seventh question). This conjecture has been proved, so we can conclude that e, pi are transcendental
numbers.

**Math Crisis**

While I was searching on transcendental numbers, I was my first time hearing about mathematical cri-
sis, that we only believe that the world has a rational number instead of irrational numbers exists back
in 18 or 19th century. The second at that time was still relatively intuitive. In fact, the transcendental
numbers basically had the same negative experience, probably a little better when people still want to
know whether all of the real numbers are algebraic - that is, are the roots of a polynomial polynomial,
whether there can not be as the real number of roots of a polynomial - this number is called the trans-
scendental number. The French mathematician Liouville gave a reply, the Liouville theorem. Definition:
If real number \( x \) satisfies, there exists positive integers \( p, q \) for any positive integer \( n \) such that:

\[
0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}
\]

\( x \) is the Liouville number.

Theorem: All Liouville numbers are transcendental numbers.

The proof of this theorem is not complicated. Subsequently, Liouville constructed the first evidenced
transcendental number according to this:
\[ 10^{-1!} + 10^{-2!} + 10^{-3!} + 10^{-4!} + \ldots \]

It is not difficult to prove that \( c \) is Liouville number at the same time. So we know exactly, there exist some strange transcendental numbers in real numbers selection. And also they will not meet any algebraic equation. It can be seen that the Liouville theorem implies that the Liouville number can be estimated with a reasonably accurate estimate by the rational number, but in fact it can be shown that the Liouville number is only a very small part of the transcendental number. Of course, only this result we are are not satisfied with those results. About a few years later, a landmark mathematician, Cantor appeared. Cantor gave us a new perspective to see some mathematical facts, creating a new Branch - set theory. Cantor also uses the set theory perspective to look at the problem of transcendence. First, he uses the diagonal method to prove that the real number is not countable - more accurate, is equal to the power of integer. I have seen this proof mentioned in the mathematical analysis textbook.

Then what about the number of algebra? In this way, for polynomials of degree \( n \), because there are only countable integers, such polynomials can only be listed, and the algebraic basic theorem guarantees that each polynomial has at most \( n \) roots, that is, such polynomials can only be listed in the number of all the roots. And then consider all the polynomials, according to the number of classification can only be classified, can be listed more than the set can still be listed, so that we proved a conclusion - algebraic number only up to more than can be counted. At the end, the conclusion is shocking, the algebraic number and real number is not even at the same level, and transcendent numbers not only exist, but also its amount is equal to real number. In other words, transcendental numbers have a big portion in real numbers.

This proof was not recognized at the time, because it is a non-constructive proof. Liouville at least created a transcendental number. However, it was hard to accept Cantor’s theory since he actually didn’t have any direct proof. It was like on a hot day, Liouville sweated through a pile of wheat, and finally found a needle, but Cantor called the weight of the wheat heap, and then asserted that the wheat mixed with the needle ... Of course, Cantor’s theory was accepted at the end. After this, people more focus on whether we can prove that some of our common numbers are transcendental numbers, like \( e \) and \( \pi \). Those are two simplest example.

Then it is important to mention the following two theorem:
- Lindemann Weierstrass theorem
- Gelfond Schneider theorem

**Algebraic and transcendental numbers**
A complex number is called algebraic if it is the root of a nonzero polynomial with integer coefficients. A complex number is called transcendental if it is not algebraic.

If \( a \) is an algebraic number and \( n \) is the smallest natural number for which there exists a polynomial with integer coefficients of degree \( n \) that has \( a \) as its root, then \( n \) is called the degree of \( a \).

For example, the number \( \sqrt{2} \) is algebraic since it is a root of the polynomial \( x^2 - 2 \). Also, its degree is 2.

Given any algebraic number \( a \), there exists a unique polynomial with rational coefficients and leading coefficient 1, that has \( a \) as its root and has minimal degree. This polynomial is called the minimal polynomial of \( a \).

**What is countable set**

Countability: A set is countable if it is finite or there is a one-to-one correspondence with the natural numbers.

A, B is called equivalent (or have the same cardinality). If there is a function \( f: A \to B \), which is 1-1.

If two sets are ”of the same cardinality”, that means that their elements can be paired off one-by-one against each other. As soon as we start doing this with a few different sets, we see that not all infinite sets are ”of the same cardinality” in this sense. For example, \( \mathbb{R} \) and \( \mathbb{N} \) cannot be paired off one-by-one against each other. In other words, there are different sizes of infinity.

The word ”countable” just means that a set is either finite, or is of the smallest type of infinity (like \( \mathbb{N} \)). It’s ”uncountable” if it’s a larger infinity - that is, it’s too big to be paired off one-by-one against \( \mathbb{N} \).

Suppose that there us \( f: A \to B \), 1-1 and \( g: B \to A \) 1-1, then \( A \) and \( B \) are equivalent.

Suppose that \( f: A=\mathbb{A} \) is then \( A \) is equivalent to \( f(A) \)

A set \( A \) is said to be finite, if \( A \) is empty or there is \( n = \mathbb{N} \) and there is a bijection \( f : 1, ..., n \to A \). Otherwise the set \( A \) is called infinite. Two sets \( A \) and \( B \) are called equinumerous, written \( A \sim B \), if there is a bijection \( f : X \to Y \). A set \( A \) is called countably infinite if \( A = \mathbb{N} \). We say that \( A \) is countable if \( A = \mathbb{N} \) or \( A \) is finite.

Tool: Bijections

Bijection from of a set \( S \): Let \( A \) and \( B \) be sets. A bijection from \( A \) to \( B \) is a function \( f : A \to B \) that is both injective and surjective.

Some properties of bijections:

Inverse functions: The inverse function of a bijection is a bijection.

Compositions: The composition of bijections is a bijection.

Basic Definition
A map \( f \) between sets \( S_1 \) and \( S_2 \) is called a bijection if \( f \) is one-to-one and onto. In other words, if \( f(a) = f(b) \) then \( a = b \). This holds for all \( a, b \) belongs to \( S_1 \).

For each \( b \in S_2 \), there is some \( a \) in \( S_1 \) such that \( f(a) = b \).

We write \( S_1 \sim S_2 \) if there is a bijection \( f : S_1 \rightarrow S_2 \). We say that \( S_1 \) and \( S_2 \) are equivalent or have the same cardinality if \( S_1 \sim S_2 \). This notion of equivalence has several basic properties:

1. \( S \sim S \) for any set \( S \). The identity map serves as a bijection from \( S \) to itself.
2. If \( S_1 \sim S_2 \) then \( S_2 \sim S_1 \). If \( f : S_1 \rightarrow S_2 \) is a bijection then the inverse map \( f^{-1} \) is a bijection from \( S_2 \) to \( S_1 \).
3. If \( S_1 \sim S_2 \) and \( S_2 \sim S_3 \) then \( S_1 \sim S_3 \). This boils down to the fact that the composition of two bijections is also a bijection.

These three properties make into an equivalence relation.

Let \( N = 1, 2, 3... \) denote the natural numbers. A set \( S \) is called countable if \( S \sim T \) for some \( T \in N \).

Here is a basic result about countable sets.

**Algebraic Number is countable**

The set of integers is countable, we have this following theorem:

Let \( A \) be a countable set, and let \( B \) be the set of \((a_1, a_2, ..., a_n)\), where \( a_k \) belongs to \( A \), \( k = 1, ..., n \), and the elements \( a_1, ..., a_n \), need to be distinct. Then \( B \) is countable.

So by this theorem, the set of all \((k+1) - (a_0, a_1, ..., a_k)\) with \( a_0 \neq 0 \) is countable as well.

Let this set be represented by the polynomial \( a_0z^k + a_1z^{k-1} + ... + a_k = 0 \)

From the fundamental theorem of algebra, we know that there are exactly \( k \) complex roots for this polynomial.

As a result, we have a countable number of \( Z^k s \), each of which corresponds with \( k \) roots of a \( k \)-degree polynomial. So the set of complex roots (call it \( R \)) is a countable union of countable unions of finite sets.

**Real Number is uncountable**

For this part, I will use cantor’s diagonal argument.

The original proof of Cantor indicates that the number of points in interval \([0,1] \) is not countable. The proof is completed by the reverse thinking, the steps are as follows:

Let’s assumed that the number of points in interval \([0,1] \) is countably infinite (derived from the original problem) So we can put all the figures in this interval into a series, \((r_1, r_2, r_3, ...)\)

Let’s assumed that the number of points in interval \([0,1] \) is countably infinite (derived from the original
problem) So we can put all the figures in this interval into a series, \((r_1, r_2, r_3, \ldots)\). It is known that each of these numbers can be expressed as a decimal numbers. In fact, some countable sets, such as rational numbers can not be in accordance with the size of the number of them in order. There are a number of forms of expression, such as \(0.499 = 0.500\), we choose the first one. If the series of decimal forms are as follows:

\[
\begin{align*}
r_1 &= 0.5105110 \\
r_2 &= 0.4132043 \\
r_3 &= 0.8245026 \\
r_4 &= 0.2330126 \\
r_5 &= 0.4107246 \\
r_6 &= 0.9937838 \\
r_7 &= 0.0105135
\end{align*}
\]

Here is a graph to explain why named diagonal argument. Pay attention to the bold number.

Let’s set a real number \(x\), where \(x\) is defined in the following manner: If the \(k\)th decimal place of \(r_k\) is equal to 5, then the \(k\)th decimal place of \(x\) is 4. If the \(k\)th decimal place of \(r_k\) is not equal to 5, the \(k\)th decimal place of \(x\) is 58. Obviously \(x\) is a real number in the interval \([0, 1]\), with the previous number as an example, the corresponding \(x\) should be \(0.4555554\).
However, because of the special definition of \( x \), this makes the \( n \)th decimal places of \( x \) and \( r_n \) different, so \( x \) does not belong to \((r_1, r_2, r_3, ...)\). Thus, \((r_1, r_2, r_3, ...)\) does not list the real numbers in all intervals \([0, 1]\), which is contradictory. Therefore, the assumption in the first point that "the number of points in interval \([0,1]\) is countable infinity" is not established.

**The uncountablity of transcendental number**

In general, since algebra number is countable, then transcendental number is uncountable.